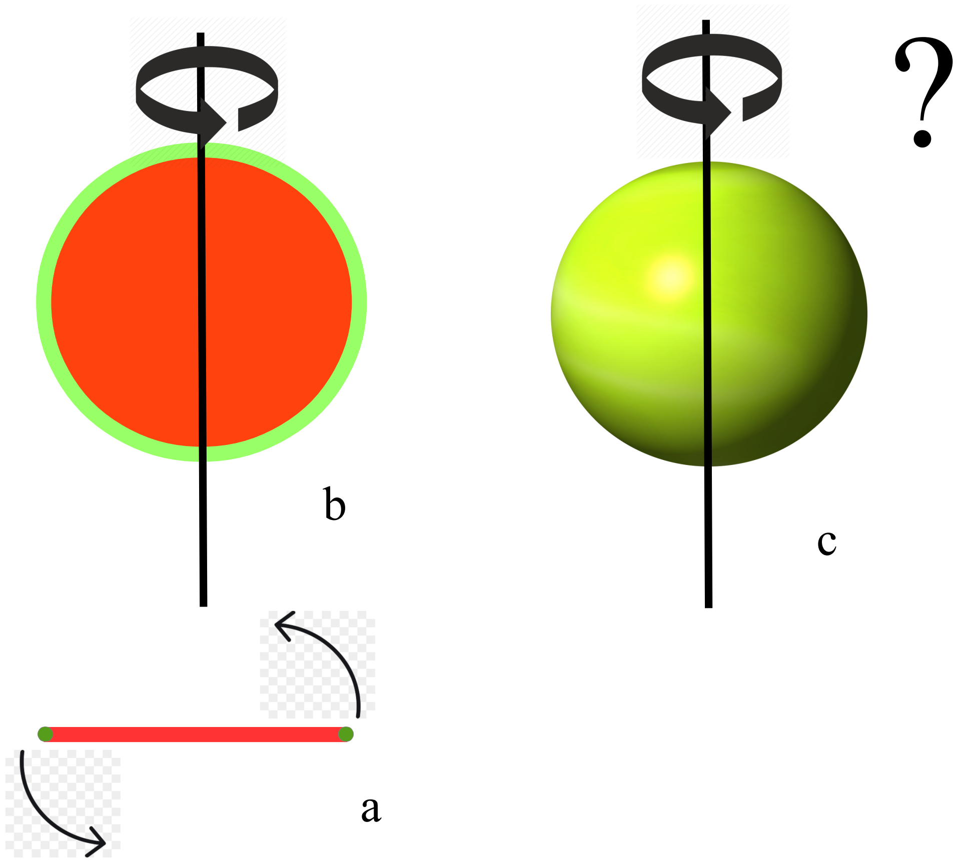
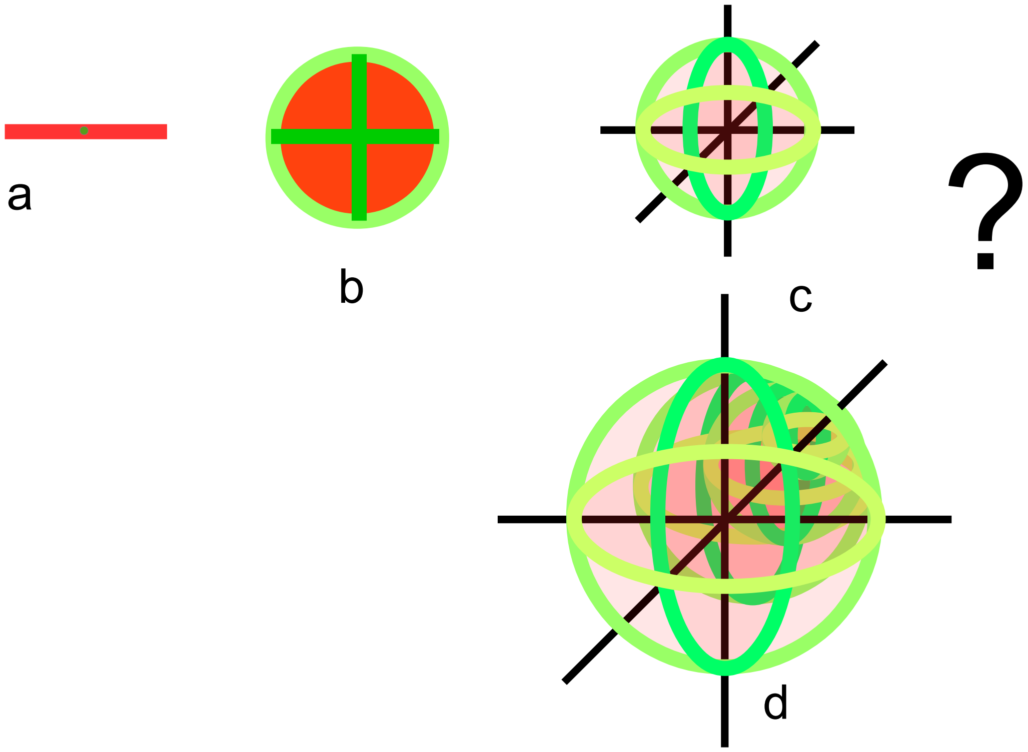
Some thoughts on a 3-sphere.

I am going to start by showing some ways that you might build up towards a 3-sphere before showing some images made by others and end with a video of a 'supercube' 3D print.

In the diagram below, 'a' is a line segment, shown in red, with its end, boundary or surface, the two green points. If this line segment is rotated through 180 degrees it traces out a 2-dimensional disc, while the zero-dimensional points trace out a circle or 1-sphere 'b'. If the disc is now rotated it traces out a solid 3-dimensional ball, while the circular 1-dimensional boundary traces out the spherical boundary, a 2-dimensional 2-sphere, of the ball. So what happens if you rotate the 2-sphere. Obviously, if you rotate it around any of the x, y or z axes, nothing happens but if it is rotated around the missing axis, the t axis, it will generate a 3-sphere, or at least some significant part of it.



Succesive rotations are one way of seeing how you might move towards a 3-sphere. Another is to look at the number and type of objects contained within objects. In the next diagram you can see that a line segment (a) (1-dimensional) can contain a single zero-dimensional point at its centre. Adisc (b) (2-dimensional) can contain two 1-dimensional lines centred at the discs centre, while a ball, (c) (3-dimensional) contains three 2-dimensional discs centred at the centre of the ball. By extrapolation it can be seen that a 3-sphere will contain four 2-spheres centred at the centre of the 3-sphere. This can also be seen from the maths.



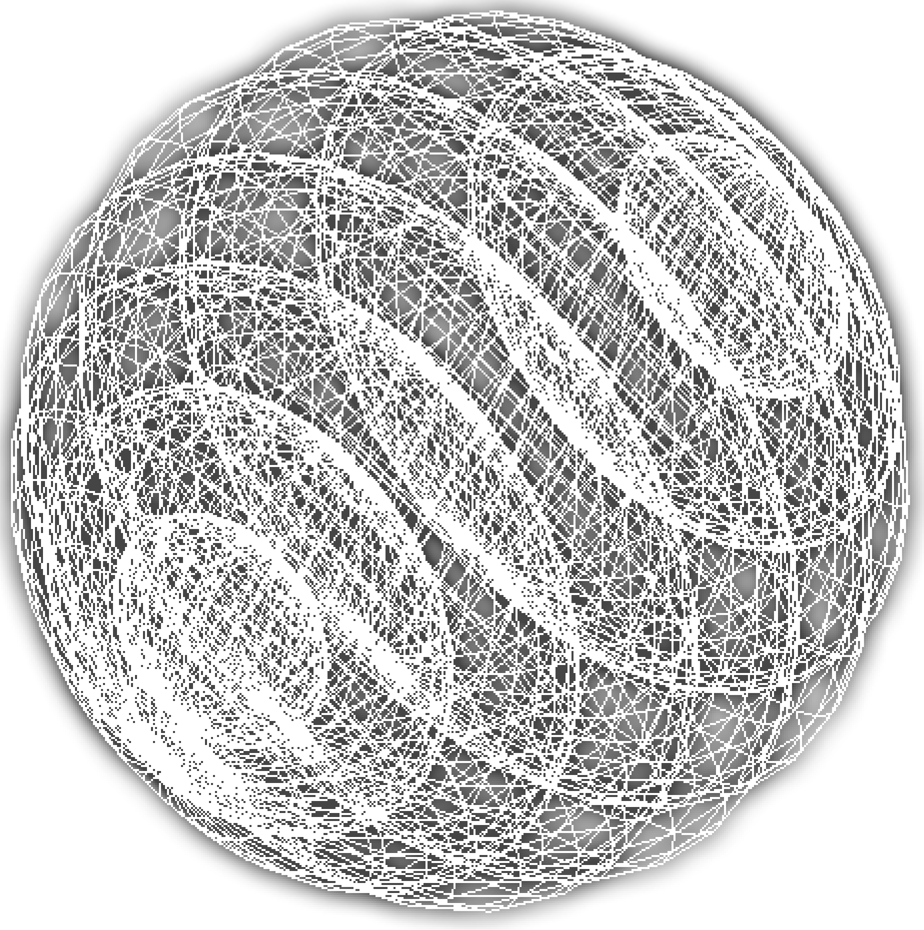
If you remember, the formula for a 3-sphere is simply

$$x^2 + y^2 + z^2 + t^2 = r^2$$

If t is set to zero then the formula for a 2-sphere results, in our every day x, y and z dimensions. But if x, y and z are in turn set to zero, instead of t, we get three more 2-spheres (x,y and t), (x, z and t) and (y, z and t). A 3-sphere contains 4 unit 2-spheres.

Diagram (d) may be a bit of a red herring, as well as being a bit messy. This shows what happens as you shift the centre of a 2-sphere along one of the axes, in this case the z axis, while at the same time making the sphere proportionally smaller. That is when the centre has moved by one quarter of the radius, the size of the sphere is reduced by 25%. when the centre has moved by one half of the radius, the size of the sphere is reduced by 50% and so on. When this is done, the shrinking 2-sphere traces out a 3-dimensional ball. But you can see from the 3-sphere formula that if you increase t from 0 to r, you must reduce the radius of the x,y,z sphere . Moving the centre of a shrinking x,y,z sphere along the z-axis traces out a 3-dimensional ball. Moving the centre of a shrinking x,y,z sphere along the t-axis traces out a 3-sphere. Actually this may just be part of the 3-sphere. It may also be necessary to move the centre of a shrinking x,y,t sphere along the z-axis and so on for the other two spheres.

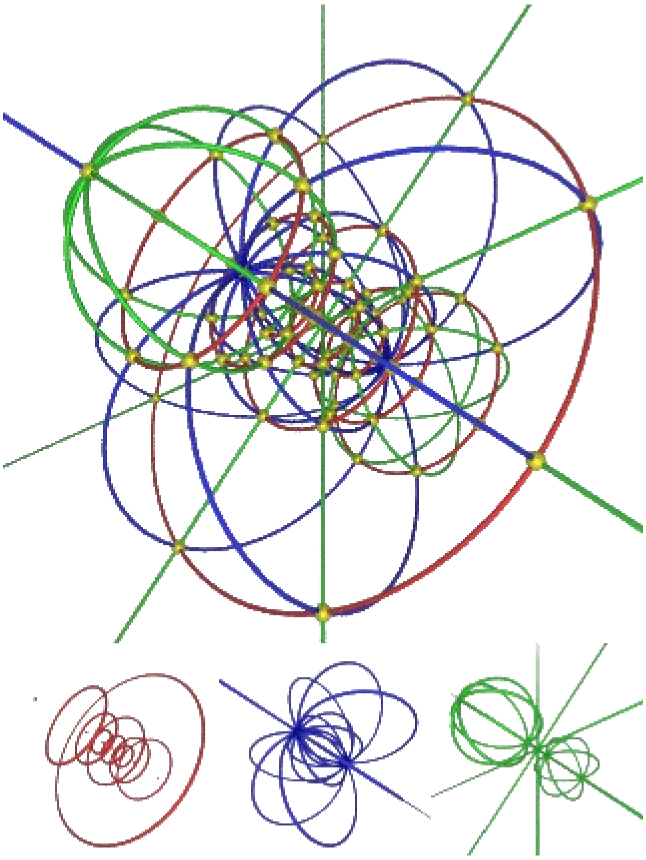
Here are two pictures of 3-spheres created by others, both taken from Wikipedia.



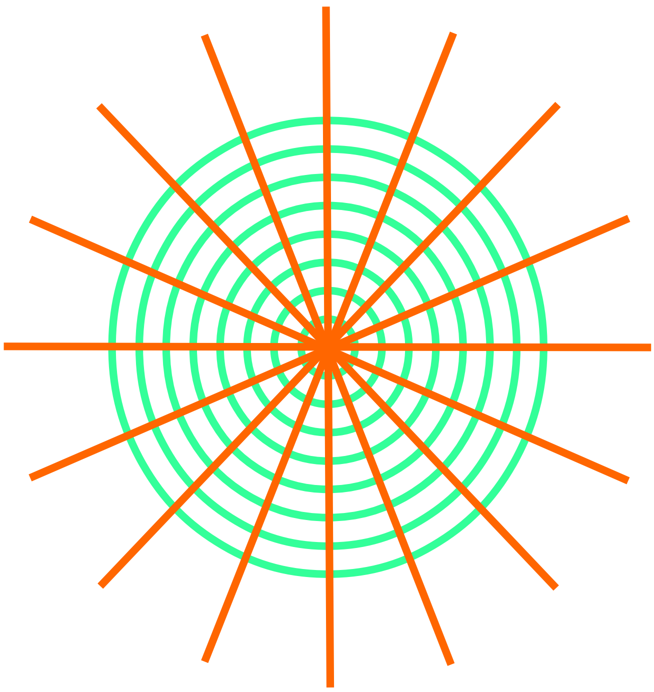
This is a direct projection of a 3-sphere into 3D space, though of course that result must then be projected onto the 2D page. It does bear some resemblance to figure (d) above.

The next image is a stereoscopic projection of a 3-sphere onto the plane. The red circles are projections of parallels, equivalent to lines of latitude on a 2-sphere, the blue lines are projections of meridians, equivalent to lines of longitude , while the green lines are projections of hypermeridians (well it was never going to be that simple). Look up 'stereoscopic projection' if you like but just for information I have added the stereoscopic projection of a 2 sphere. The orange lines are projections of lines of longitude, while the green circles are projections of lines of latitude. Both sets of lines continue to infinity. And while it adds little to this section, here also is a stereoscopic projection of the northern hemisphere, made from the south pole. If this image were extended to include the southern hemisphere, the size of countries would balloon in size, very rapidly. The south pole itself lies at infinity. It is best to make a stereoscopic projection of the southern hemisphere from the north pole.

Stereoscopic projection of a 3-sphere.



Stereoscopic projection of a 2-sphere.



Stereoscopic projection of the northern hemisphere, from the south pole.

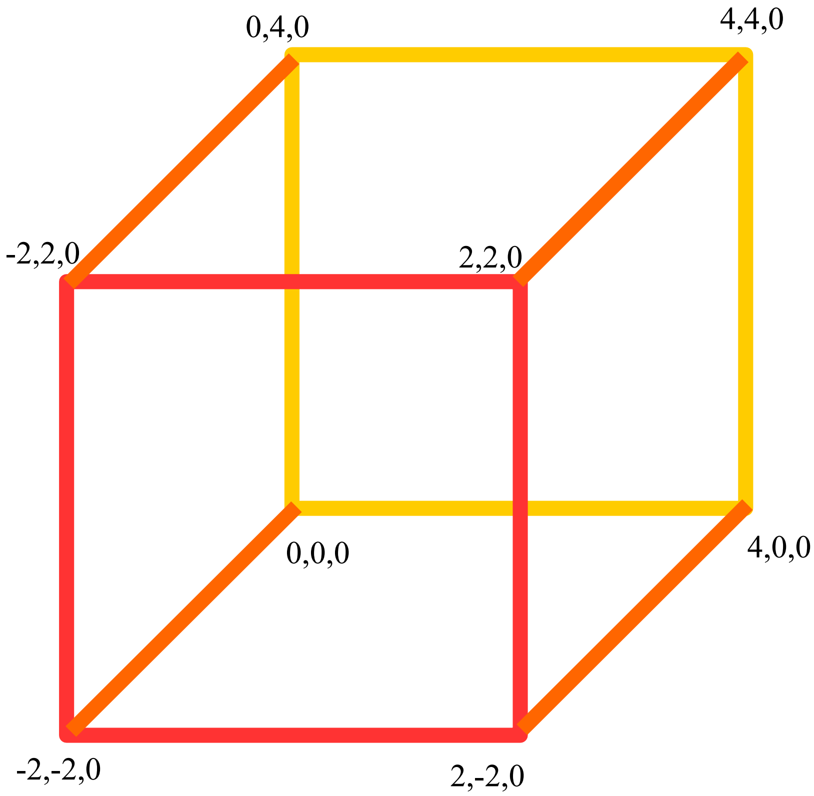
Back to 3-spheres

As I said above, the unit 3-sphere contains four unit 2-spheres, each obtained by setting one of the four co-ordinates, x, y, z and t, to zero. As you can see, it is very difficult to depict, let alone imagine these spheres. There is,however, something interesting that can be done. A cube is topologically the same as a sphere. If that doesn't make much sense, a cube can always be created from eight points suitable drawn on a sphere. A unit 3-sphere contains four unit cubes. One of these is the x,y,z cube with the following co-ordintes.

1,1,1,0
1,-1,1,0
1,1,-1,0
1,-1,-1,0
-1,-1,-1,0
-1,-1,1,0
-1,1,-1,0
-1,1,1,0

These co-ordinates could be used to draw a cube on a sheet of paper or to make a 3D model of a cube. The other three cubes can be generated by swapping the final column of zeros, in turn with each of the other three columns. These three cubes can only be depicted in four dimensions, which might seem difficult, but there is a method. This method will be explained by considering how we draw a cube on a 2-dimensional piece of paper. This is something that most of us can do just by instinct. However, there is also a procedure for doing this, known as isometric projection. Once this understood it becomes quite simple to project a 4-dimensional object into three dimensions, and then to use a 3D printer to print it out.

So let us start with a square, the red square below.



The co-ordinates of this red square are shown, that in the top left being -2,2,0. While we are sticking to two dimensions we could get rid of the third dimension, whose numbers are all zero anyway. The length of a side of this square is 4 units. We can create a cube by adding 4 units to the z dimension to create a second square. This square has the following co-ordinates

-2,2,4
2,2,4
-2,-2,4
2,-2,4

We can use these co-ordinates, together with the four co-ordinates of red square, making eight vertices in total, to create a cube that can be built in three dimensions. But this cannot be shown in two dimensions because the second square lies directly under the first. The trick is as follows. Take the z value for any point. Divide it into 2 and add each half to one of the other two dimensions and then set the z value to zero. So -2,2,4 becomes -2 + 4/2, 2 + 4/2, 0 or 0,4,0. If this is done for each of the four co-ordinates of the new square we get

0,4,0
4,4,0
0,0,0
4,0,0

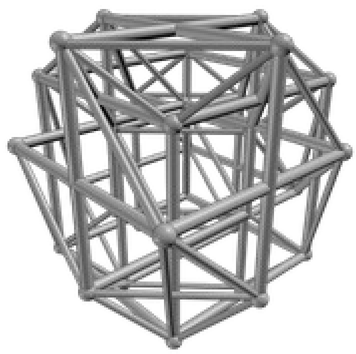
When these new points are plotted on our sheet of paper, we get the yellow square. It is now a simple matter to connect the two squares, to create the impression of a cube in two dimensions. This is shown by adding the orange lines. The technique is known as an isometric projection. This might seem a fairly laborious procedure for something that comes naturally to many people. The reason for explaining it in detail is that the procedure can easily be extended to more dimensions, but stopping at four in our case. Take any co-ordinate in four dimensions, such as

$$x, y, z, t$$

By using the same technique we can create a 3-dimensional co-ordinate

$$x + t/3,\ y + t/3,\ z + t/3,\ 0$$

So the point 1, 1, 0, 1 will become 1.33, 1.33, 0.33, 0. I carried out this procedure for all of the 32 co-ordinates of the four unit cubes in a 3-sphere, and then connected the co-ordinates to create an isometric projection of these four cubes into our three dimensions. The aim was to print out these cubes using a 3D printer. However, before doing this I noticed that the same 32 vertices could be connected to each other in a different way to create four additional cubes. Each vertex was now connected to three other vertices in its own cube and to the corresponding vertex of each of the other three unit cubes. The result is shown below. The edges of the original four unit cubes are slightly thicker than the edges of the second set of four cubes.



Of course, this is a 2D representation of the 3D model. Every vertex has six edges, but many are obscured. I can't see any vertex with six edges in this diagram. To get a better effect, I have made a video of the model which can be seen by returning to the blog.